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Correlation Inequalities and the Kosterlitz–Thouless Transition for Anisotropic Rotators

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I give a proof of the Kosterlitz–Thouless transition for sufficiently anisotropic ($J_z J_x^{-1} = J_z J_y^{-1} < 2q^{-1}(J_{\mathrm{KT}})^{-1}$) two-dimensional N-component rotators ($N \geqslant 3$). The method is based on Wells' inequality and is related to mean field Gaussian inequalities.

KEY WORDS: Anisotropic rotators; Kosterlitz–Thouless phase; mean field inequalities.

1. LOWER BOUND AND THE KOSTERLITZ–THOULESS TRANSITION

In a recent paper,[(1)] Bricmont, Lebowitz, and Pfister prove, among other things, the occurrence of the Kosterlitz–Thouless transition for the "infinitely" anisotropic ($J_z = 0$, $J_x = J_y$) two-dimensional classical Heisenberg model. The authors use Wells' inequality,[(12)] which says that a one-component classical spin model $\{\sigma_i\}_{i \in \mathbb{Z}^d}$, with ferromagnetic couplings $\{J_{ij}\}_{i,j}$ and with single spin distribution not concentrated on $\sigma_i = 0$,

$$d\nu(\sigma_i) \neq \delta(\sigma_i)$$

has correlation functions bounded below by the correlations of an Ising model:

$$\langle \sigma_A \rangle_{\nu, \{J_{ij}\}_{ij}} \geqslant \langle \sigma_A \rangle_{\pm 1, \{a^2 J_{ij}\}_{ij}}$$

The scaling factor a^2 is independent of the couplings, and long-range order for the Ising model implies long-range order for the model with single spin distribution ν.

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Bricmont, Lebowitz, and Pfister[12] remark that the same comparison can be made between two-component rotator models with single spin distribution, respectively,

$$d\nu(|\boldsymbol{\sigma}|)\,d\varphi$$

and

$$\delta(|\boldsymbol{\sigma}|^2 - 1)\,d\varphi$$

The classical Heisenberg model has a single spin distribution

$$\sin \vartheta\, d\vartheta\, d\varphi = d\nu(\sin \vartheta)\, d\varphi$$

and can be considered, for $J_z = 0$, as a model of plane rotators of random lengths $\{\sin \vartheta_i\}_i$. The occurrence of the Kosterlitz–Thouless transition for this model then follows from the results[4] of Fröhlich and Spencer for unit rotators.

The same remark applies to $P(|\boldsymbol{\Phi}|^2)$ models defined on the lattice $\mathbb{Z}^2$, when $\boldsymbol{\Phi}$ has two components, but the estimates vanish in the continuum limit.

Let us note finally that Wells' inequality does not extend to models with more than two components (the Ginibre inequality[5] is a crucial ingredient).

The N-component rotator model on $\mathbb{Z}^d$ is a model of random unit vectors ("classical spins")

$$\mathbf{S}(j) \in \mathbb{R}^N, \qquad |\mathbf{S}(j)| = 1, \qquad j \in \mathbb{Z}^d$$

In order to compare $N > 2$ with $N = 2$, we introduce the notation

$$\mathbf{S}(j) = (\boldsymbol{\sigma}(j), \boldsymbol{\tau}(j)) \in \mathbb{R}^2 \times \mathbb{R}^{N-2} \tag{1}$$

The couplings will satisfy the associated $O(2) \times O(N-2)$ invariance: the $O(2)$ invariance in view of the desired Kosterlitz–Thouless transition, and the $O(N-2)$ invariance to simplify the notation.

The joint probability distribution, or Boltzmann factor, is chosen as

$$\begin{aligned}
Z^{-1} \exp &\left\{\sum_{i,j} [J_{ij}\boldsymbol{\sigma}(i)\cdot\boldsymbol{\sigma}(j) + K_{ij}\boldsymbol{\tau}(i)\cdot\boldsymbol{\tau}(j)]\right\} \prod_j \delta(\mathbf{S}(j)^2 - 1)\, d^N\mathbf{S}_j \\
&= Z^{-1} \exp\left[\sum_{i,j} (J_{ij} \sin \vartheta_i \sin \vartheta_j \cos(\varphi_i - \varphi_j) \right. \\
&\quad \left. + K_{ij} \cos \vartheta_i \cos \vartheta_j \hat{\tau}(i)\cdot\hat{\tau}(j))\right] \\
&\quad \times \prod_j d\varphi_j \sin \vartheta_j (\cos \vartheta_j)^{N-3}\, d\vartheta_j\, d\hat{\tau}(j)
\end{aligned} \tag{2}$$

where

$$\hat{\tau}(j) = |\tau(j)|^{-1}\, \tau(j)$$

Theorem 1. Let $\langle \cdot \rangle_{J,K}^{(N)}$ denote the expectation with respect to (2) and let $\langle \cdot \rangle_J^{(2)}$ denote the analogous expectation for $N=2$. Let

$$K = \sup_i \sum_j |K_{ij}|$$

and

$$\gamma_N(K) = \sup \left\{ \gamma \mid \gamma \leqslant 1/2 \text{ and } \int_0^1 (1-\gamma-x^2)\, e^{(1-\gamma)^{1/2} K x} x^{N-3}\, dx \geqslant 0 \right\} \tag{3}$$

Suppose that $J_{ij} \geqslant 0$ $\forall i, j$. Then for any index set A (e.g., from $\mathbb{Z}^d \times \mathbb{Z}^d$) we have

$$\left\langle \prod_{\{k,l\} \in A} (\sigma(k) \cdot \sigma(l)) \right\rangle_{J,K}^{(N)} \geqslant [\gamma_N(K)]^{|A|} \left\langle \prod_{\{k,l\} \in A} (\sigma(k) \cdot \sigma(l)) \right\rangle_{\gamma_N(K) J}^{(2)} \tag{4}$$

Moreover: $\gamma_N(K)$ is a decreasing function of both N and K,

$$K\gamma_N(K) \to 2 \text{ as } K \to \infty, \qquad \forall N \in \mathbb{N}$$

$$N = 3, \qquad K \leqslant 2 \Rightarrow \gamma_3(K) = 1/2$$

Remark. Theorem 1 proves what is asserted in the abstract. Consider an N-component model on a two-dimensional lattice, where each spin is coupled with q neighbors, with coupling constants $\beta J_x = \beta J_y$ for the first two components, and in modulus at most βJ_z for each of the remaining $(N-2)$ components. Suppose that

$$J_z J_x^{-1} < 2q^{-1} (J_{\mathrm{KT}})^{-1}$$

where J_{KT} is the inverse temperature (including β) at which the Kosterlitz–Thouless transition occurs for plane rotators ($N=2$). Then

$$\gamma_N(q\beta J_z)\, \beta J_x \to 2q^{-1} J_x J_z^{-1} > J_{\mathrm{KT}} \qquad \text{as} \quad \beta \to \infty$$

and the correlations of the first two components of the N-component rotator are bounded below by the correlations of a plane rotator in the Kosterlitz–Thouless phase.

Note however that for all $J_z < J_x = J_y$, the correlation functions of the z components are expected[1] to decay exponentially at all temperatures (in any dimension). The existence of the Kosterlitz–Thouless phase is also expected for all $J_z < J_x = J_y$ and β sufficiently large. On the contrary, for $J_z = J_x = J_y$, all correlations are expected to decay exponentially (in two dimensions). If we accept this last point, our inequalities can be considered as mean field bounds for the plane rotator model. Indeed (4) can be written as

$$\langle \boldsymbol{\sigma}(k) \cdot \boldsymbol{\sigma}(l) \rangle^{(2)}_{\gamma_N(qJ)J} \leqslant \gamma_N(qJ)^{-1} \langle \boldsymbol{\sigma}(k) \cdot \boldsymbol{\sigma}(l) \rangle^{(N)}_{J,J} \tag{5}$$

Since

$$\gamma_N(qJ)J \to 2q^{-1} \qquad \text{as} \quad J \to \infty$$

and assuming that the right-hand side of (5) decays exponentially, we conclude

$$J_{\mathrm{KT}} \geqslant 2q^{-1}$$

which is the mean field bound. A more natural approach to mean field bounds will be described in Section 2.

Proof of Theorem 1. For A a set of indices from $\mathbb{Z}^d$ and $m(.)$ a function from $\mathbb{Z}^d$ to $\mathbb{Z}$, we denote

$$(\sin \vartheta)_A \cos m \cdot \varphi = \prod_{j \in A} (\sin \vartheta_j) \cos \left[\sum_j m(j)\, \varphi_j \right]$$

We then have

$$\begin{aligned} Z^{(N)}_{J,K} &\langle (\sin \vartheta)_A \cos m \cdot \varphi \rangle^{(N)}_{J,K} \\ &= \int \prod_j d\hat{\tau}(j)\, Z^{(3')}_{J,K(\hat{\tau})} \langle (\sin \vartheta)_A \cos m \cdot \varphi \rangle^{3'}_{J,K(\hat{\tau})} \\ &\geqslant \int \prod_j d\tau(j)\, Z^{(3')}_{J,K(\hat{\tau})} \langle (\sin \vartheta)_A \cos m \cdot \varphi \rangle^{3'}_{J,|K(1)|} \\ &= Z^{(N)}_{J,K} \langle (\sin \vartheta)_A \cos m \cdot \varphi \rangle^{3'}_{J,|K(1)|} \end{aligned} \tag{6}$$

where $\langle . \rangle^{3'}_{J,K(\hat{\tau})}$ is the expectation with respect to (2) conditioned by given values of $\{\hat{\tau}(j)\}_j$. We use the symbol 3′ to indicate that the individual distribution

$$d\varphi_j \sin \vartheta_j (\cos \vartheta_j)^{N-3}\, d\vartheta_j$$

is not isotropic in $\mathbb{R}^3$ (except for $N=3$), and that ϑ_j varies only from 0 to $\pi/2$. We have

$$|K(1)_{ij}| = |K_{ij}|$$

and the inequality (6) comes from.[2,6]

We now have to compare

$$\langle \cdot \rangle^{3'}_{J,|K(1)|} \quad \text{and} \quad \langle \cdot \rangle^{(2)}_{\gamma J}$$

and we introduce a joint probability distribution

$$Z^{-1}Z'^{-1} \exp\Big(\sum_{i,j} \{J_{ij}[\sin\vartheta_i \sin\vartheta_j \cos(\varphi_i - \varphi_j) + \gamma\cos(\varphi'_i - \varphi'_j)] + |K_{ij}| \cos\vartheta_i \cos\vartheta_j\}\Big) \times \prod_j d\varphi'_j \, d\varphi_j \sin\vartheta_j (\cos\vartheta_j)^{N-3} \, d\vartheta_j \tag{7}$$

Denoting the corresponding expectation by $\langle\langle \cdot \rangle\rangle$, we have

$$\langle (\sin\vartheta)_A \cos m\cdot\varphi \rangle^{3'}_{J,|K(1)|} - \gamma^{|A|/2} \langle \cos m\cdot\varphi \rangle_{\gamma J} = \langle\langle (\sin\vartheta)_A \cos m\cdot\varphi - \gamma^{|A|/2} \cos m\cdot\varphi' \rangle\rangle \tag{8}$$

In order to apply the method of Wells' inequality, we write

$$\cos\vartheta_i \cos\vartheta_j = (\gamma' - \cos\vartheta_i)(\gamma' - \cos\vartheta_j) + \gamma'\cos\vartheta_i + \gamma'\cos\vartheta_j - \gamma'^2$$

The advantage of this formula is that (e.g., $\gamma'=1$) the coupling term

$$|K_{ij}| \, (\gamma' - \cos\vartheta_i)(\gamma' - \cos\vartheta_j)$$

now favors $|\sin\vartheta_i| = 1$, which favors the existence of the Kosterlitz–Thouless phase. This is important because coupling terms, as opposed to single site terms, must be expanded in most proofs of correlation inequalities.

The idea that a coupling term (here $K_{ij}\cos\vartheta_i\cos\vartheta_j$), which is unfavorable for some behavior of the model, can sometimes be compensated for by single site terms [here $-K_{ij}(\cos\vartheta_i + \cos\vartheta_j)$] was used by the author in Ref. 3 for the gauge invariant Ising model (four-body plaquette couplings go against the Lee–Yang theorem) and for the Widom–Rowlinson model.

We now expand the Boltzmann factor (7) and the integrand in (8) as a multinomial with positive coefficients in terms of

$$\cos(\varphi_i - \varphi_j) \pm \cos(\varphi'_i - \varphi'_j)$$

$$\sin \vartheta_j \pm \gamma^{1/2}, \qquad \gamma' - \cos \vartheta_j$$

$$\exp\left(\gamma' \cos \vartheta_j \sum_i |K_{ij}|\right)$$

The integral over $\{\varphi_j\}_j$ and $\{\varphi'_j\}_j$ is positive by the Ginibre inequality. The remaining integral factorizes over the sites and we only have to show

$$\sum_{\substack{\sigma=\pm 1\\ \sigma'=\pm 1}} \int_0^{\pi/2} (\sigma \sin \vartheta + \sigma' \gamma^{1/2})^p (\sigma \sin \vartheta - \sigma' \gamma^{1/2})^q (\gamma' - \cos \vartheta)^r \times e^{\gamma' K \cos \vartheta} \sin \vartheta (\cos \vartheta)^{N-3} \, d\vartheta \geqslant 0 \tag{9}$$

with

$$K = \sup_i \sum_j |K_{ij}|$$

and where σ and σ' have been saved (as redundant variables) from φ and φ': a random vector **s** with an even probability distribution, such as the uniform measure on the circle, can be replaced by σ**s** with $\sigma = \pm 1$ (arbitrary weights) and the unchanged distribution for **s**. The point is that the $U(1)$ symmetry cannot be conveniently factorized over sites, whereas it is possible, and useful, to keep the spin flip symmetry, included in the $U(1)$ symmetry, when factorizing over sites.

We may now suppose $p \geqslant q$, $p - q$ even, so that

$$\sum_{\substack{\sigma=\pm 1\\ \sigma'=\pm 1}} (\sigma \sin \vartheta + \sigma' \gamma^{1/2})^{p-q}$$

can be expanded with positive coefficients, in powers of $\sin^2 \vartheta$, and *a fortiori* also in powers of $\sin^2 \vartheta - \gamma$. Condition (9) with

$$\gamma' = (1-\gamma)^{1/2}$$

then reduces to the condition

$$\int_0^{\pi/2} (\sin^2 \vartheta - \gamma)^n \, e^{(1-\gamma)^{1/2} K \cos \vartheta} \sin \vartheta (\cos \vartheta)^{N-3} \, d\vartheta \geqslant 0 \tag{10}$$

Taking $n=1$ and $n=\infty$ gives hypothesis (3) in Theorem 1. Monotonicity of $\gamma_N(K)$ in N and K can be considered as a correlation inequality in the measure (10) (take the integrand with $n=0$): $\sin^2\vartheta-\gamma$ is a decreasing function of $\cos\vartheta$, whereas K and N come in front of increasing functions of $\cos\vartheta$. Therefore[5] the normalized expectation

$$\langle \sin^2\vartheta-\gamma\rangle_{K,N}$$

is a decreasing function of K and N.

To see that (10) is also satisfied for $1<n<\infty$, we write it as follows (n odd, $0<\gamma\leqslant 1/2$):

$$\int_0^{1-\gamma} t^n\big(e^{\gamma' K(1-\gamma-t)^{1/2}}(1-\gamma-t)^{N/2-2} - \vartheta(\gamma-t)\, e^{\gamma' K(1-\gamma+t)^{1/2}}(1-\gamma+t)^{N/2-2}\big)\, dt$$

where

$$\vartheta(\gamma-t)=\begin{cases}1, & \text{if } t<\gamma\\ 0, & \text{if } t>\gamma\end{cases}$$

The integrand is positive for $t>\gamma$ and changes sign at most once in $]0, 1-\gamma]$, say in τ. It follows that the integral with $n\in\mathbb{N}$ is bigger than or equal to τ^{n-1} times the integral with $n=1$, and is therefore positive.

This concludes the proof of Theorem 1. ∎

2. UPPER BOUND: MEAN FIELD GAUSSIAN INEQUALITIES

Theorem 2. Let $\langle\,.\,\rangle_{J,K}^{(N)}$ denote the expectation with respect to (2). Suppose that $K_{ij}\geqslant 0$ and $J_{ii}=0$ $\forall i,j$ and that

$$\sup_i \sum_j |J_{ij}| < N$$

Let $\langle\,.\,\rangle_G$ be the two component $O(2)$ symmetric massive Gaussian model of inverse covariance

$$(C^{-1})_{ij} = -|J_{ij}| + N\delta_{i,j}$$

where $\delta_{i,j}$ is the Kronecker symbol (e.g., on $\mathbb{Z}^d\times\mathbb{Z}^d$).

For any index set A (e.g., from $\mathbb{Z}^d\times\mathbb{Z}^d$), we have

$$\Big\langle \prod_{\{k,l\}\in A} (\boldsymbol{\sigma}(k)\cdot\boldsymbol{\sigma}(l))\Big\rangle_{J,K}^{(N)} \leqslant \Big\langle \prod_{\{k,l\}\in A} (\boldsymbol{\sigma}(k)\cdot\boldsymbol{\sigma}(l))\Big\rangle_G$$

Remarks. (i) Theorem 2 was proved by Tasaki[11] using a high-temperature expansion. See also Refs. 7–10 for previous results and further references.

(ii) The condition $J_{ii}=0$ is not essential, but simplifies the formulation.

(iii) Theorem 2 can be extended to include an external field and give mean field upper bounds on the magnetization.

Lemma 1. Let $\langle\,.\,\rangle_{J,K}^{(N)}$ denote the expectation with respect to (2). Suppose that $K_{ij}\geqslant 0\ \forall i, j$. Then

$$\left\langle \prod_{\{k,l\}\in A} (\boldsymbol{\sigma}(k)\cdot\boldsymbol{\sigma}(l)) \right\rangle_{J,K}^{(N)} \leqslant \left\langle \prod_{\{k,l\}\in A} (\boldsymbol{\sigma}(k)\cdot\boldsymbol{\sigma}(l)) \right\rangle_{|J|,0}^{(N)}$$

Proof of Lemma 1. Lemma 1 is a variant of Theorem 6 in Ref. 7, where Pearce decomposes $\mathbb{R}^N=\mathbb{R}\times\mathbb{R}^{N-1}$. I choose $\mathbb{R}^N=\mathbb{R}^2\times\mathbb{R}^{N-2}$ for technical convenience, but mean field bounds should be the same in both ways. The proof of Lemma 1 is essentially identical to Pearce's.

Proof of Theorem 2. We first use Lemma 1 and then consider, for $J_{ij}\geqslant 0$, the joint probability distribution

$$Z^{-1}Z'^{-1}\exp\left\{\sum J_{ij}[\sin\vartheta_i\sin\vartheta_j\cos(\varphi_i-\varphi_j)+\rho_i\rho_j\cos(\varphi'_i-\varphi'_j)]\right\}$$
$$\times\prod_i \sin\vartheta_i(\cos\vartheta_i)^{N-3}\,d\vartheta_i\,d\varphi_i\,e^{-N/2\,\rho_i^2}\rho_i\,d\rho_i\,d\varphi'_i$$

Denoting the corresponding expectation by $\langle\langle\,.\,\rangle\rangle$, we should prove

$$\left\langle\!\!\left\langle \prod_{\{k,l\}\in A}(\rho_k\rho_l\cos(\varphi'_k-\varphi'_l)-\sin\vartheta_k\sin\vartheta_l\cos(\varphi_k-\varphi_l)) \right\rangle\!\!\right\rangle\geqslant 0$$

We expand the integrand and the Boltzmann factor in terms of

$$\rho_i\pm\sin\vartheta_i$$
$$\cos m\cdot\varphi\pm\cos m\cdot\varphi'$$

and the problem reduces to proving, for all p and q,

$$\sum_{\sigma,\sigma'}\int_0^{\pi/2}\sin\vartheta(\cos\vartheta)^{N-3}\,d\vartheta$$
$$\times\int_0^\infty e^{-N/2\,\rho^2}\rho\,d\rho(\sigma\rho+\sigma'\sin\vartheta)^p(\sigma\rho-\sigma'\sin\vartheta)^q\overset{?}{\gtrless}0 \qquad (11)$$

where σ and σ' are again redundant variables which were added to φ and φ'. By spin flip symmetry, we may suppose $p \geqslant q$ and $p-q$ even, so that

$$(\sigma\rho + \sigma' \sin \vartheta)^{p-q} = (\rho^2 + \sin^2 \vartheta + 2\sigma\sigma'\rho \sin \vartheta)^{p-q/2}$$

and (11) now reduces to

$$\int_0^{\pi/2} \sin \vartheta (\cos \vartheta)^{N-3}\, d\vartheta \int_0^\infty e^{-N\rho^2/2} \rho\, d\rho (\rho^2 - \sin^2 \vartheta)^n \times (\rho^2 + \sin^2 \vartheta)^u (\rho^2 \sin^2 \vartheta)^v \overset{?}{\gtrless} 0$$

The Gaussian measure was chosen so that the integral vanishes for $n=1$, $u=v=0$. The only difficulty is to prove that this is the most restrictive case. For this purpose, we introduce the change of variables

$$|\rho^2 - \sin^2 \vartheta| = t$$
$$\cos^2 \vartheta = y$$

and the integral now reads (n odd)

$$\int_0^\infty dt\, t^n \left[\int_0^1 dy\, y^{N/2-2} e^{-N/2(t+1-y)} (t+2-2y)^u (t+1-y)^v (1-y)^v - \vartheta(1-t) \int_0^{\max\{0,1-t\}} dy\, y^{N/2-2} e^{-N/2(-t+1-y)} (-t+2-2y)^u \times (-t+1-y)^v (1-y)^v \right] \overset{?}{\gtrless} 0 \tag{12}$$

For $n=1$ and $u=v=0$, we have

$$\int_0^\infty dt\, t \left[\int_0^1 dy\, y^{N/2-2} e^{-N(t+1-y)/2} - \vartheta(1-t) \int_0^{1-t} dy\, y^{N/2-2} e^{-N(-t+1-y)/2} \right] = 0 \tag{13}$$

Lemma 2. Let

$$I_N(s) = \int_0^s dy\, y^{N/2-2} e^{N/2\, y}$$

and

$$f_N(s) = N(1-s) + \log I_N(s) - \log I_N(1)$$

The function $f_N(s)$ vanishes at most once in $]0, 1[$.

Proof of Lemma 2. We compute

$$f_N''(s) = -I_N(s)^{-2}\, s^{N/2-3} e^{N/2\, s} \left[s^{N/2-1} e^{N/2\, s} - I_N(s) \left(\frac{N}{2} - 2 + \frac{N}{2} s \right) \right]$$

The bracket equals

$$s^{N/2-1} \left(\frac{N}{2} - 1 \right)^{-1} + \sum_{p=1}^{\infty} \frac{(N/2)^p\, s^{N/2-1+p}}{p!} \left(1 - \frac{N/2-2}{N/2-1+p} - \frac{p}{N/2-2+p} \right) \tag{14}$$

which is positive term by term for $N \geqslant 4$. In this case the function $f_N(s)$ is concave on $]0, 1[$ and vanishes at $s = 1$, which implies the lemma.

For $N = 3$, the sum from $p = 1$ to ∞ in (14) is negative term by term. It follows that $f_3''(s)$ vanishes at most once in $]0, 1[$. We now remark that $f_3(s)$ is negative for $s \to 0$ and positive for $s \to 1$.

The number of zeros in $]0, 1[$ must therefore be odd. With only one inflexion point, there can be only one zero, which concludes the proof of the lemma. ▮

Proof of Theorem 2. The bracket in (13) is positive for $t \geqslant 1$, and zero at $t = 0$. Lemma 2 implies that it vanishes exactly once in $]0, 1]$, say in t_N. This implies, for all $t_1 < \infty$,

$$\int_0^{t_1} dt\, t \int_0^1 dy\, y^{N/2-2} e^{-N(t+1-y)/2}$$

$$< \int_0^{t_1} dt'\, t' \vartheta(1-t') \int_0^{1-t'} dy'\, y'^{N/2-2} e^{-N(-t'+1-y')/2}$$

From this we conclude that we can find a diffeomorphism between the integration domains

$$\begin{aligned} y' &= y'(t, y) = [1 - t'(t, y)]\, y \\ t' &= t'(t, y) \end{aligned} \tag{15}$$

with

$$t'(t, y) \leqslant t \tag{16}$$

and

$$t y^{N/2-2} e^{-N(t+1-y)/2} = (dt'/dt)\, t' (dy'/dy)\, y'^{N/2-2} e^{-N(-t'+1-y')/2}$$

If we now put back into (13) the factors

$$t^{n-1}(t+2-2y)^u(t+1-y)^v(1-y)^v$$

and

$$t'^{n-1}(-t'+2-2y')^u(-t'+1-y')^v(1-y')^v$$

the desired inequalities (12) follow from the inequalities

$$t \geqslant t'$$
$$t+2-2y \geqslant -t'+2-2y'$$
$$(t+1-y)(1-y) \geqslant (-t'+1-y')(1-y')$$

which are easily checked from (15), (16).

This concludes the proof of Theorem 2. ∎

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